The equation $x - 3y - 4z = 0$ describes a plane in $\mathbb{R}^3$.

(a) The plane is the nullspace of some matrix. What is it?

Solution.

Rewriting the given equation in matrix form

$$\begin{pmatrix} 1 - 3 - 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and we readily see that every point on the plane is in the nullspace of

$$A = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}.$$

(b) Find a basis for the nullspace.

Solution.

We form the matrix

$$N = \begin{pmatrix} 3 & 4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So the nullspace is spanned by $(3, 1, 0)^T$ and $(4, 0, 1)^T$.

(c) Find a basis for the line $P^{\perp}$.

Solution.

This is the normal vector given by the coefficients of the equation $(1, -3, -4)^T$.

(d) Decompose $v = (6, 4, 5)^T$ into its nullspace component and a component in $P^{\perp}$.

Solution.

We need to solve

$$\alpha \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}.$$

This is the column version of the following matrix multiplication

$$\begin{pmatrix} 1 & 3 & 4 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$$

which one can solve through row reductions of the augmented matrix. We get

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Our decomposition is as follows:

$$\begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}}_{\text{particular}} + \underbrace{\begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}}_{\text{special}}$$

Suppose S is a collection of vectors, not necessarily a subspace. Show that $S^{\perp}$ is a subspace.

Proof. Let x, y be two arbitrary vectors perpendicular to S and α some real number. Then we need to show that

$$(x + \alpha y)^T s = 0 \text{ for every } s \in S.$$

The above is a matrix multiplication. Then,

$$(x^T + \alpha y^T)s = x^T s + \alpha y^T s = 0 + 0 = 0$$

which is obtained via the distributive law for matrix multiplication. We conclude that $(x + \alpha y)^T$ also lies in $S^{\perp}$. □